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On the stability of the unsteady boundary layer on a cylinder oscillating transversely in a viscous fluid

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The stability of the two-dimensional flow induced by the transverse oscillation of a cylinder in a viscous fluid is investigated in both the linear and weakly nonlinear regimes. The major assumption that is made to simplify the problem is that the oscillation frequency is large, in which case an unsteady boundary layer is set up on the cylinder. The basic flow induced by the motion of the cylinder depends on two spatial variables, and is periodic in time. The stability analysis of this flow to axially periodic disturbances therefore leads to a partial differential system dependent on three variables. In the high-frequency limit the linear stability problem can be reduced to a system dependent only on a radial variable and time. Furthermore, the coefficients of the differential operators in this system are periodic in time, so that Floquet theory can be used to reduce this system further to a coupled infinite system of ordinary differential equations together with uncoupled homogeneous boundary conditions. The eigenvalues of this system are found numerically and predict instability entirely consistent with the experiments with circular cylinders performed by Honji (1981). Results are given for cylinders of elliptic cross-section, and it is found that for any given eccentricity the most dangerous configuration is when the cylinder oscillates parallel to its minor axis. Some discussion of nonlinear effects is also given, and for the circular cylinder it is shown that the steady-streaming boundary layer of the basic flow is significantly altered by the instability.

1. Introduction

Our concern is with the stability of a class of flows that exhibit the phenomenon usually referred to as 'steady streaming'. In particular, motivated by the recent experiments of Honji (1981), we consider in detail the stability of the flow induced by the transverse oscillations of a circular cylinder of radius a in a viscous fluid of kinematic viscosity ν. This flow has been investigated by several authors following the boundary-layer approach used by Schlichting (1932). For a detailed discussion of the steady streaming induced by the oscillation of the cylinder the reader is referred to the papers of Stuart (1966) and Riley (1967).

The experiments of Honji illustrated clearly a phenomenon surprisingly not reported in previous experimental investigations of the flow. We refer to the observation made by Honji that the two-dimensional flow induced by the motion of the cylinder is unstable to axially periodic vortices of the Taylor–Görtler type at sufficiently large values of the amplitude of oscillation of the cylinder. The instability occurs in the Stokes layers at the cylinder in the locations where they are parallel to the direction of motion of the cylinder. The instability is apparently of the centrifugal type, and is initially in the form of vortices aligned with the local flow direction. However, the steady streaming associated with the basic flow convects the dye used to visualize the vortices away from the Stokes layer. At larger amplitudes of oscillation the dye streaks produced by the vortices disappear, and the flow is said by Honji to be turbulent and separated. It was suggested by Honji that the instability might be of the type that is known to occur in a Stokes layer on a torsionally oscillating cylinder, and it is this possibility that we shall investigate in this paper.

This latter type of instability has been investigated in detail by Seminara & Hall (1976, 1977), Park, Barenghi & Donnelly (1980) and Hall (1981). Suppose then that an infinitely long cylinder oscillates torsionally about its axis with angular velocity $\Omega \cos \omega t$ in a viscous fluid. At sufficiently small values of ω the flow is purely circumferential, whilst if ω is slowly increased then at a critical value of $\omega = \omega_{c1}$ an array of vortices, periodic along the cylinders, develops in the boundary layer at the cylinder. The strength of the vortices increases as ω is increased further, but at a second critical value of ω, say ω_{c2}, the vortices interact with each other, and the flow rapidly becomes turbulent with no apparent periodicity along the cylinders. The theoretical description of this flow for $\omega < \omega_{c2}$ given by Seminara & Hall (1976, 1977) was verified experimentally by Park *et al.* (1980). However, the secondary stage of the instability is perhaps only partially explained by the subharmonic instability mechanism described by Hall (1981). It is of interest to note that the values of ω_{c1} and ω_{c2} are quite close, so that, once the initial vortex structure has been set up, a relatively small increase in ω leads to a turbulent flow. Thus, if this instability mechanism is indeed operating in the experiments of Honji, it is clear that the two-dimensional flows of the type discussed by Schlichting (1932), Stuart (1966), etc. will be greatly altered. The primary aim of the present investigation is to determine the parameter range in which the two-dimensional solution is a stable solution of the Navier–Stokes equations. The feature of the basic flow that makes a stability calculation non-trivial is, of course, the fact that the basic flow depends upon time and two spatial coordinates.

It is to be expected that at sufficiently large amplitudes of oscillation the boundary layer at the cylinder will separate and the attached-flow strategy of the type discussed by Schlichting and subsequent authors fails. The results of Honji suggest that this does not occur before the instability mechanism is operational. Thus our stability calculation provides an upper limit for the oscillation amplitude beyond which there is no reason to compute the basic flow. It is therefore possible that any laminar separation theory for the oscillating cylinder problem will not be relevant to experimental observations.

Suppose then that a circular cylinder of radius a oscillates with velocity $U_0 \cos \omega t$ along a diameter in a fluid of viscosity ν. The parameters that govern the two-dimensional flow are

$$\beta = \frac{\omega a^2}{\nu}, \quad \lambda = \frac{U_0}{\omega a}, \quad R_s = \frac{U_0^2}{\omega \nu} = \lambda^2 \beta. \tag{1.1 a, b, c}$$

The frequency parameter β is taken to be large, so that the unsteady boundary layer on the cylinder is thin compared with its radius. The parameter λ represents the ratio of the amplitude of oscillation of the cylinder to the cylinder radius and is taken to be small. Stuart (1966) has discussed the crucial role played by the steady-streaming Reynolds number R_s in determining the nature of the steady streaming set up outside the Stokes layer on the cylinder. If R_s is small the motion is determined by solving the Stokes equations, whereas for large R_s an outer boundary-layer flow exists.

In order to obtain some idea about the parameter regime in which an instability might occur we note that the first-order oscillatory flow set up by the motion of the cylinder is confined to a thin layer of thickness $(\nu/\omega)^{\frac{1}{2}}$ at the cylinder, and so the radius of curvature of the paths of fluid particles is of order a. Thus the Taylor number that characterizes this boundary-layer flow is of order $U_0^2/a\nu^{\frac{1}{2}}\omega^{\frac{3}{2}} = R_s\beta^{-\frac{1}{2}}$. The instability mechanism described by Seminara & Hall (1976, 1977) operates when this Taylor number is $O(1)$), so we conclude that in the present problem the regime of interest is $R_s \sim \beta^{\frac{1}{2}}$. For this reason we confine our attention in this paper to the stability of the two-dimensional flow around the cylinder in the limit $\beta \to \infty$, $R_s = O(\beta^{\frac{1}{2}})$. We further note that in this limit λ is $O(\beta^{-\frac{1}{4}})$, so that the boundary layer on the cylinder is essentially a Stokes layer.

The above comparison between the torsionally and transversely oscillating-cylinder flows ignores the spatial variation around the cylinder of the first-order boundary-layer flow in the latter case. An examination of this structure shows that the flow is locally most unstable at the positions $\theta = \pm\frac{1}{2}\pi$ if the direction of oscillation is along the x-axis. We shall show that a self-consistent asymptotic description of the linear stability problem is possible for $R_s \sim \beta^{\frac{1}{2}}$, $\beta \to \infty$. Furthermore, we show that the instability is confined to $\beta^{-\frac{1}{8}}$ neighbourhoods of the positions $\theta = \pm\frac{1}{2}\pi$. More precisely we show that the flow is formally unstable when

$$R_s > R_{sc} = R_0\beta^{\frac{1}{2}} + R_1\beta^{\frac{1}{4}} + R_2\beta^{\frac{1}{8}} + \dots, \tag{1.2}$$

where R_0, R_1, etc. are $O(1)$ constants to be evaluated. In fact, we determine only R_0 and R_1, and find that the resulting critical value of λ agrees almost exactly with the experimental results of Honji.

The analysis used for the circular-cylinder problem can be easily modified to more complicated steady-streaming flows. We shall show that, having investigated the linear stability problem for the circular cylinder we can, to first order in β, write down the critical stability parameter with only a knowledge of the first-order outer potential flow. However, at the next order there are technical differences between the circular-cylinder problem and for example the problem associated with elliptic cylinders. We shall see that these technical differences depend on whether or not the stagnation point of attachment of the steady streaming coincides with the most unstable part of the boundary layer.

Some discussion of nonlinear effects for $R_s - R_{sc} - O(\beta^{\frac{1}{4}})$ is given. For the circular-cylinder problem a strong interaction between the steady streaming and the instability occurs. In fact it appears that the higher modes of instability lead to the separation of the steady-streaming boundary layer within an angle $O(\beta^{-\frac{1}{8}})$ of the point of attachment of the layer.

The procedure adopted in the rest of this paper is as follows. In §2 the linear stability problem is formulated for $\beta \to \infty$, $R_s \sim \beta^{\frac{1}{2}}$, and an asymptotic solution of the problem is given. In §3 the results of the numerical solution of the eigenvalue problem obtained in §2 are given and compared to Honji's experimental observation.

In §4 we discuss the relevance of our calculations to more complicated flows. More precisely we consider the stability of the flow induced by the oscillation of an elliptic cylinder. We consider an ellipse with major and minor axis a and b with major axis inclined at angle α to the x-axis, in which direction the cylinder is oscillating. We find that, depending on the values of α and b/a, there are two or six locations where instability will occur. In §5 we consider the nonlinear development of the instability, whilst in §6 we draw some conclusions.

2. Formulation and solution of the linear stability problem in the limit $\beta \to \infty$

The first step in our formulation is to note that by a simple change of axes we can take the cylinder to be held fixed whilst the fluid at infinity oscillates with speed $U_0 \cos \omega t$ parallel to the x-axis. It is convenient for us to work in cylindrical polar coordinates (r, θ, z') with the z' axis along the axis of the cylinder. We now define the variables η, z and τ by

$$\eta = \left(\frac{r}{a} - 1\right)\left(\frac{\beta}{2}\right)^{\frac{1}{2}}, \quad z = \frac{z'}{a}\left(\frac{\beta}{2}\right)^{\frac{1}{2}}, \quad \tau = \omega t. \tag{2.1a, b, c}$$

Following the scalings discussed in §1, we write

$$R_s = \frac{T\beta^{\frac{1}{2}}}{2^{\frac{3}{2}}}, \tag{2.2}$$

where T is $O(\beta^0)$ and is, of course, the Taylor number. We shall investigate the stability of the boundary layer on the cylinder in which the basic velocity field is $(\bar{u}, \bar{v}, 0)$, with

$$\bar{u} = (2\nu\omega)^{\frac{1}{2}} \left\{ \frac{\cos\theta\, \bar{u}_0(\eta, \tau)}{\beta^{\frac{1}{4}}} + \ldots \right\}, \tag{2.3a}$$

$$\bar{v} = 2U_0 \left\{ \sin\theta\, \bar{v}_0(\eta, \tau) + \frac{\sin 2\theta}{\beta^{\frac{1}{4}}} \bar{v}_1(\eta, \tau) + \ldots \right\}, \tag{2.3b}$$

where in particular

$$\bar{v}_0 = \cos\tau - \cos(\tau - \eta)\, \mathrm{e}^{-\eta},$$

while $\bar{u}_0$, $\bar{v}_1$, etc. can be found in, for example, Stuart (1966).

Following the scalings used by Seminara & Hall (1976), we perturb the basic flow such that the new velocity field is

$$\boldsymbol{u} = (\bar{u}, \bar{v}, 0) + ((2\nu\omega)^{\frac{1}{2}}\, U(\eta, \theta, z, \tau),\ U_0 V(\eta, \theta, z, \tau),\ (2\nu\omega)^{\frac{1}{2}}\, W(\eta, \theta, z, t)), \tag{2.4}$$

whilst the corresponding pressure perturbation is $\rho\omega\nu P(\eta, \theta, z, \tau)$. If the above expression is substituted into the Navier–Stokes equations we find that U, V, W, and P satisfy

$$\mathrm{L}'U = \frac{\partial P}{\partial \eta} - 2T \sin\theta\, \bar{v}_0\, V + Q_1 + O(\beta^{-\frac{1}{4}}), \tag{2.5a}$$

$$\mathrm{L}'V = 2^{\frac{3}{8}} T^{-\frac{1}{2}} \beta^{-\frac{3}{4}} \frac{\partial P}{\partial \theta} + 4 \sin\theta \frac{\partial \bar{v}_0}{\partial \eta} U + Q_2 + O(\beta^{-\frac{1}{4}}), \tag{2.5b}$$

$$\mathrm{L}'W = \frac{\partial P}{\partial z} + Q_3 + O(\beta^{-\frac{1}{4}}), \quad \frac{\partial U}{\partial \eta} + \frac{2^{-\frac{3}{4}} T^{\frac{1}{2}}}{\beta^{\frac{1}{4}}} \frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0. \tag{2.5c, d}$$

Here the nonlinear terms Q_1, Q_2 and Q_3 are given by

$$Q_1 = 2\left(U\frac{\partial U}{\partial \eta} + W\frac{\partial U}{\partial z}\right) - \frac{T}{2} V^2, \quad Q_2 = 2\left(U\frac{\partial V}{\partial \eta} + W\frac{\partial V}{\partial z}\right), \quad Q_3 = 2\left(U\frac{\partial W}{\partial \eta} + W\frac{\partial W}{\partial z}\right),$$

whilst the operator L' has been defined by

$$\mathrm{L}' \equiv \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial z^2} - 2\frac{\partial}{\partial \tau} - \frac{2^{\frac{5}{4}} T^{\frac{1}{2}} \sin\theta\, \bar{v}_0}{\beta^{\frac{1}{4}}} \frac{\partial}{\partial \theta}. \tag{2.6}$$

We further note that the $O(\beta^{-\frac{1}{4}})$ terms not shown explicitly in (2.5) comprise both linear and nonlinear terms. However, the linear terms not shown explicitly vanish when $\theta = \pm\frac{1}{2}\pi$, and for that reason are negligible in the following analysis. The nonlinear terms not shown explicitly do not vanish at $\theta = \pm\frac{1}{2}\pi$, but the smallness of the disturbance which we assume in §5 means that, to the order considered in this paper, these terms are also negligible.

For the remainder of this section we neglect the nonlinear terms in (2.5) and assume that P, U and V are proportional to $\cos kz$ whilst W is proportional to $\sin kz$. Here k is a constant axial wavenumber, and it is now convenient to eliminate W and P from the linearized form of (2.5) to give

$$\mathrm{L}\left(\frac{\partial^2}{\partial\eta^2}-k^2\right)U = 2k^2 T\sin\theta\,\bar{v}_0\,V-\frac{2^{\frac{5}{4}}\sin\theta\,\bar{v}_{0\eta\eta}T^{\frac{1}{2}}}{\beta^{\frac{1}{4}}}\frac{\partial U}{\partial\theta}+O(\beta^{-\frac{1}{4}}), \tag{2.7a}$$

$$\mathrm{L}V = 4\sin\theta\frac{\partial\bar{v}_0}{\partial\eta}U+O(\beta^{-\frac{1}{4}}), \tag{2.7b}$$

which are to be solved subject to

$$U = V = \frac{\partial U}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0,$$

$$U, V \to 0 \quad \text{as} \quad \eta\to\infty.$$

The operator L appearing in (2.7) is simply L′ with $\partial^2/\partial z^2$ replaced by $-k^2$.

It can be seen from (2.7) that the θ-variation of the disturbance is slow compared with the τ- and η-variations. The θ-dependence of U and V can therefore be taken care of by a WKB type of approach. However, since we are interested in the most unstable disturbances, it is convenient for us to use a multiple-scale method. We can see from (2.7) that, ignoring the term proportional to $\beta^{-\frac{1}{4}}\partial/\partial\theta$, the 'effective' Taylor number of the flow is $T\sin^2\theta$, which has local maxima at $\theta = \pm\frac{1}{2}\pi$. Hence in the neighbourhood of say $\theta = \frac{1}{2}\pi$ the effective Taylor number is $T\{1-(\theta-\frac{1}{2}\pi)^2+\dots\}$. The symmetry of (2.7) about $\theta = \frac{1}{2}\pi$ means that when the WKB formulation is used the point $\theta = \frac{1}{2}\pi$ is a turning point. Since the local Taylor number has a maximum at $\theta = \frac{1}{2}\pi$ the turning point is of the second order and the usual scaling analysis shows that a transition layer of thickness $O(\beta^{-\frac{1}{8}})$ exists near $\theta = \frac{1}{2}\pi$. This situation is similar to that found by Hall (1982), who investigated the growth of small-wavelength Görtler vortices in boundary layers on concave walls. In that problem the most unstable modes have a vertical structure concentrated in an internal transition layer, which again corresponds to a second-order turning point.

The discussion above clearly also applies to the neighbourhood of $\theta = -\frac{1}{2}\pi$, but let us concentrate on the transition layer at $\theta = \frac{1}{2}\pi$ and write

$$\Phi = (\theta-\tfrac{1}{2}\pi)\,\beta^{\frac{1}{8}}.$$

We seek a solution of (2.7) by expanding U in the form

$$U = U_0(\eta,\tau,\Phi)+\beta^{-\frac{1}{8}}U_1(\eta,\tau,\Phi)+\beta^{-\frac{1}{4}}U_2(\eta,\tau,\Phi)+\dots, \tag{2.8}$$

together with a similar expansion for V. The Taylor number T then expands as

$$T = T_0+\beta^{-\frac{1}{4}}T_1+\beta^{-\frac{3}{8}}T_2+\dots. \tag{2.9}$$

Here we have anticipated that the first-order correction to T from T_0 is $O(\beta^{-\frac{1}{4}})$ rather than $O(\beta^{-\frac{1}{8}})$ as might be expected from (2.8). The above expansions are then substituted into (2.7) and θ replaced by $\frac{1}{2}\pi+\beta^{-\frac{1}{8}}\Phi$. If terms $O(\beta^0)$ are equated we obtain

$$\left.\begin{aligned}\left\{\frac{\partial^2}{\partial\eta^2}-k^2-2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2}-k^2\right\}U_0 &= 2k^2\bar{v}_0 T_0 V_0,\\ \left\{\frac{\partial^2}{\partial\eta^2}-k^2-2\frac{\partial}{\partial\tau}\right\}V_0 &= 4\frac{\partial\bar{v}_0}{\partial\eta}U_0,\end{aligned}\right\} \tag{2.10}$$

and the appropriate boundary conditions are

$$\left.\begin{aligned}U_0 = \frac{\partial U_0}{\partial\eta} = V_0 = 0 \quad \text{at} \quad \eta = 0,\\ U_0, V_0 \to 0 \quad \text{as} \quad \eta\to\infty.\end{aligned}\right\} \tag{2.11}$$

The partial differential system (2.10), (2.11) governs the centrifugal instability of a Stokes layer on a cylinder driven by a pressure gradient rather than by the motion of the cylinder as was the case in Seminara & Hall (1976). The major difference is that in the latter paper the function $\bar{v}_0$ is replaced by $\cos(\tau-\eta)\,\mathrm{e}^{-\eta}$. If we were concerned with the instability of flat Stokes layers driven by a pressure gradient or a moving wall we could show the equivalence of the corresponding instability problems by a simple change of axis. In the present situation there is no such transformation, and so the eigenvalues of (2.10) differ from those of Seminara & Hall (1976). In order to determine the value of $T_0 = T_0(k)$ above which exponentially growing solutions of (2.10) and (2.11) exist we seek periodic solutions of the latter system by writing

$$U_0 = A(\Phi)\sum_{-\infty}^{\infty} U_0^n(\eta)\,\mathrm{e}^{\mathrm{i}n\tau}, \quad V_0 = A(\Phi)\sum_{-\infty}^{\infty} V_0^n(\eta)\,\mathrm{e}^{\mathrm{i}n\tau}.$$

The sequences of functions $\{U_0^n\}$ and $\{V_0^n\}$ therefore satisfy the ordinary differential system

$$\left.\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2-2\mathrm{i}n\right]\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2\right]U_0^n = k^2T_0\{[1-\mathrm{e}^{-\eta(1+\mathrm{i})}]\,V_0^{n-1}+[1-\mathrm{e}^{-\eta(1-\mathrm{i})}]\,V_0^{n+1}\},\\ &\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}-k^2-2\mathrm{i}n\right]V_0^n = -2\{(1+\mathrm{i})\,\mathrm{e}^{-\eta(1+\mathrm{i})}\,U_0^{n-1}+(1-\mathrm{i})\,\mathrm{e}^{-\eta(-\mathrm{i})}\,U_0^{n+1}\},\\ &U_0^n = \frac{\mathrm{d}}{\mathrm{d}\eta}U_0^n = V_0^n = 0 \quad \text{at} \quad \eta = 0,\\ &U_0^n, V_0^n \to 0 \quad \text{as} \quad \eta\to\infty, \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots.\end{aligned}\right\} \tag{2.12}$$

The numerical solution of (2.12) will be discussed later, and it suffices to say that the eigenrelation $k = k(T_0)$ can be determined. The amplitude function $A(\Phi)$ remains undetermined at this order.

At order $\beta^{-\frac{1}{8}}$ the partial differential equations satisfied by U_1 and V_1 are found to be

$$\left.\begin{aligned}&\left\{\frac{\partial^2}{\partial\eta^2}-k^2-2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2}-k^2\right\}U_1-2k^2T_0\bar{v}_0 V_1\\ &\qquad = 2^{\frac{5}{4}}T_0^{\frac{1}{2}}\frac{\partial}{\partial\Phi}\left[\bar{v}_0\left(\frac{\partial^2}{\partial\eta^2}-k^2\right)U_0-\bar{v}_{0\eta\eta}U_0\right],\\ &\left\{\frac{\partial^2}{\partial\eta^2}-k^2-2\frac{\partial}{\partial\tau}\right\}V_1-4\frac{\partial\bar{v}_0}{\partial\eta}U_1 = 2^{\frac{5}{4}}T_0^{\frac{1}{2}}\bar{v}_0\frac{\partial V_0}{\partial\Phi},\end{aligned}\right\} \tag{2.13a}$$

together with boundary conditions identical with (2.11). The differential system for (U_1, V_1) is an inhomogeneous form of (2.10) and (2.11), and so we require that a solvability condition must be applied if the system is to have a solution. However, it can be inferred from (2.12) that the sequences of eigenfunctions $\{U_0^n\}$ and $\{V_0^n\}$ are such that either

$$(a) \quad U_0^n = 0 \ (n \text{ even}), \quad V_0^n = 0 \ (n \text{ odd}),$$

or

$$(b) \quad U_0^n = 0 \ (n \text{ odd}), \quad V_0^n = 0 \ (n \text{ even}).$$

In fact, our calculation showed that the most unstable mode corresponds to (*b*) above. Hence if we expand (U_1, V_1) in the form

$$U_1 = \sum_{-\infty}^{\infty} U_1^n(\Phi, \eta)\,\mathrm{e}^{\mathrm{i}n\tau}, \quad V_1 = \sum_{-\infty}^{\infty} V_1^n(\Phi, \eta)\,\mathrm{e}^{\mathrm{i}n\tau},$$

then from (2.13*a*) it follows that only the equations for U_1^n and V_1^{n+1}, for n odd, are forced. Thus no solvability condition on (2.13*a*) is needed, and the solution of the system for (U_1, V_1) can be written

$$U_1 = \frac{\mathrm{d}A}{\mathrm{d}\Phi}\hat{U}_1 + B(\Phi)\,U_0 = \frac{\mathrm{d}A}{\mathrm{d}\Phi}\sum_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\tau}U_1^n + B(\Phi)\sum_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\tau}U_0^n, \tag{2.13b}$$

together with a similar expression for V_1. Here B is another amplitude function to be determined at higher order.

At order $\beta^{-\frac{1}{4}}$ the function pair (U_2, V_2) is found to satisfy

$$\left\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - k^2\right\} U_2 - 2k^2 T_0 \bar{v}_0 V_2$$

$$= 2^{\frac{5}{4}} T_0^{\frac{1}{2}} \frac{\partial}{\partial\Phi}\left\{\bar{v}_0\left(\frac{\partial^2}{\partial\eta^2} - k^2\right) U_1 - \bar{v}_{0\eta\eta}\, U_1\right\} + 2k^2\bar{v}_0(T_1 - \tfrac{1}{2}\Phi^2 T_0)\, V_0$$

$$\left\{\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right\} V_2 - 4\frac{\partial\bar{v}_0}{\partial\eta}\, U_2 = 2^{\frac{5}{4}} T_0^{\frac{1}{2}} \bar{v}_0 \frac{\partial V_1}{\partial\Phi} - 2\frac{\partial\bar{v}_0}{\partial\eta}\, U_0\,\Phi^2,$$

whilst the boundary conditions are again identical with (2.11). The forcing terms on the right-hand sides of the above equations are synchronous with the solutions of the homogeneous forms of the equations, and a solution for (U_2, V_2) will not in general exist. However, by considering the partial differential system adjoint to (2.10) and (2.11), we find that a solution exists if

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\Phi^2} + \mu(T_1 - \Phi^2)\,A = 0, \tag{2.14}$$

where μ is given by

$$\mu = \frac{k^2 \int_0^{2\pi}\int_0^{\infty} U^+ \bar{v}_0\, V_0\,\mathrm{d}\eta\,\mathrm{d}\tau}{2^{\frac{1}{4}} T_0^{\frac{1}{2}} \int_0^{2\pi}\int_0^{\infty} \left[\bar{v}_0\, V^+ \hat{V}_1 - \bar{v}_{0\eta\eta}\, U^+ \hat{U}_1 + \bar{v}_0\, U^+\left(\frac{\partial^2}{\partial\eta^2} - k^2\right)\hat{U}_1\right]\mathrm{d}\eta\,\mathrm{d}\tau}, \tag{2.15}$$

and (U^+, V^+) satisfy the adjoint differential system

$$\left\{\frac{\partial^2}{\partial\eta^2} - k^2 + 2\frac{\partial}{\partial\tau}\right\} V^+ = 2k^2 T_0 \bar{v}_0\, U^+, \quad \left\{\frac{\partial^2}{\partial\eta^2} - k^2 + 2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - k^2\right\} U^+ = 4\bar{v}_{0\eta}\, V^+,$$

$$U^+ = \frac{\partial U^+}{\partial\eta} = V^+ = 0 \quad \text{at} \quad \eta = 0,$$

$$U^+, V^+ \to 0 \quad \text{as} \quad \eta \to \infty.$$

The eigenvalues of the adjoint system are of course identical with those of (2.10) and (2.11), and can be obtained by Fourier-expanding (U^+, V^+) to obtain an infinite set of coupled ordinary differential equations.

The constant μ is a function of k, and our computation suggests that it is positive near the critical value of T_0. This means that if we ignore the term proportional to Φ^2 in (2.14) and assume that $A \sim e^{im\Phi}$ then the flow is neutrally stable when

$$T_1 = \frac{m^2}{\mu}.$$

Thus non-axisymmetric modes are more stable than the axisymmetric mode $m = 0$. This result was found by Duck & Hall (1980) for the case when the flow is driven by the motion of the cylinder, whilst the present results show that this is also the case if a pressure gradient is driving the Stokes layer. The amplitude equation (2.14) has solutions that decay to zero when $\Phi \to \pm\infty$ if $\mu > 0$. These solutions are

$$A(\Phi) = A_n(\Phi) = U_n(-n-\tfrac{1}{2}, 2\mu^{\frac{1}{4}}\Phi), \tag{2.16}$$

where U_n is the nth parabolic-cylinder function and the value of T_1 corresponding to A_n is

$$T_1 = T_{1n} = 2\frac{n+\frac{1}{2}}{\mu^{\frac{1}{2}}}. \tag{2.17}$$

The function $A_n(\Phi)$ has $n-1$ zeros in $(-\infty, \infty)$ and is even or odd in ϕ depending on whether n is an even or odd integer. The functions all tend to zero like $\exp[-\frac{1}{2}\mu^{\frac{1}{2}}\Phi^2]$ when $|\Phi| \to \infty$, and the least stable mode corresponds to $n = 0$, in which case

$$A_0(\Phi) = \exp[-\tfrac{1}{2}\mu^{\frac{1}{2}}\Phi^2]. \tag{2.18}$$

The expansion procedure described above can be continued to any order, and we note here that the next non-zero term in the expansion of T is T_2.

3. The numerical solution of the linear eigenvalue problem

The solution of eigenvalue problems such as (2.12) is now a routine procedure, and we shall give only the essential details of the calculations. The first step is to reduce (2.12) to a finite set of equations by setting $U_n = V_n = 0$ for $|n| > M$. We then replace ∞ by η_∞, so that (2.12) has been approximated by a finite system of equations on a finite interval. Of course, it is necessary to vary η_∞ and M to find appropriate values which enable us to solve (2.12) with sufficient accuracy.

If η is sufficiently large then U_0^n and V_0^n satisfy

$$\left[\frac{d^2}{d\eta^2} - k^2 - 2in\right]\left[\frac{d^2}{d\eta^2} - k^2\right] U_0^n = k^2 T_0 [V_0^{n-1} + V_0^{n+1}],$$

$$\left[\frac{d^2}{d\eta^2} - k^2 - 2in\right] V_0^n = 0.$$

There are three independent solutions of these equations which decay exponentially to zero when $\eta \to \infty$. Thus, if n is restricted to the range $-M \leqslant n \leqslant M$ we can use these solutions to integrate the differential equations from $\eta = \eta_\infty$ to $\eta = 0$, thus obtaining $6M+3$ independent solutions of the reduced system of equations. This integration was carried out using a fourth-order Runge–Kutta scheme with steplength h. These independent solutions of (2.12) can be combined at $\eta = 0$ to satisfy $6M+2$ of the required boundary conditions there. The remaining boundary condition is

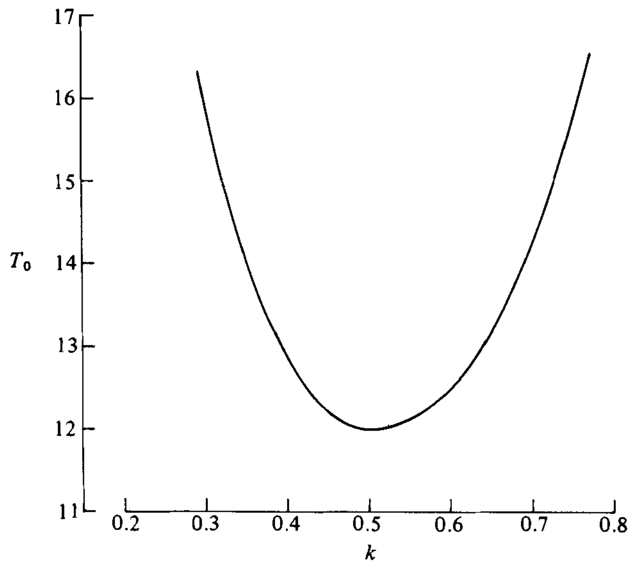

FIGURE 1. The neutral curve of the linear problem.

automatically satisfied if $k = k(T_0)$ is an eigenvalue of the reduced system. This eigenvalue of the reduced system will depend on M, η_∞ and h, but by increasing M, η_∞ and decreasing h an eigenvalue of (2.12) can be obtained.

In our calculations it was found that $M = 6$, $\eta_\infty = 10$ and $h = 0.25$ gave results correct to the accuracy given in this section. In figure 1 we show the neutral curve $k = k(T_0)$, and the minimum of this curve corresponds to

$$T_0 = T_{0c} = 11.99, \quad k = k_c = 0.51. \tag{3.1 a, b}$$

It is interesting to note that the corresponding value of T_0 for a torsionally oscillating cylinder is ≈ 230, so that transverse oscillations of the cylinder produce a much more unstable flow. There is no obvious physical reason why this should be the case.

The eigenfunctions corresponding to figure 1 were normalized by taking $U_0^{0\prime\prime}(0) = 1$ and have the property that $U_0^n = 0$ and $V_0^{n+1} = 0$ when n is an odd integer. The functions U_0^0 and V_1^1 corresponding to the critical case are shown in figure 2. We note that the disturbance is most pronounced near $\eta \approx 3$. It is interesting to note that an asymptotic solution of (2.12) in the limit $a \to \infty$ with $T \sim a^4$ shows that the vortices become concentrated in an internal viscous layer of thickness $a^{-\frac{1}{2}}$. In this layer the functions U_0^n and V_0^n can all be expressed in terms of parabolic-cylinder functions. A similar calculation for the torsionally oscillating cylinder problem shows that when $a \to \infty$, with $T \sim a^4$, the vortices become concentrated in a layer of thickness $a^{-\frac{2}{3}}$ near $\eta = 0$, and the eigenfunctions are then determined in terms of the Airy function $\mathrm{Ai}\,(x)$.

In order to check the eigenvalues shown in figure 1, the solution of the adjoint system was computed in a similar manner. The adjoint eigenfunctions were found to have the property $(U_1^n)^+$, $(V_1^{n+1})^+ = 0$ for n an odd integer. The inhomogeneous system for $\hat{U}_1$ and $\hat{V}_1$ was found by a shooting procedure similar to that used for (2.12). The integrals appearing in the definition of μ were then evaluated using Simpson's rule. We obtained $\mu = 0.033$ with T_0 and k as given by (3.1 a, b). The critical value of T_1 is therefore given by

$$T_{1c} = 5.51,$$

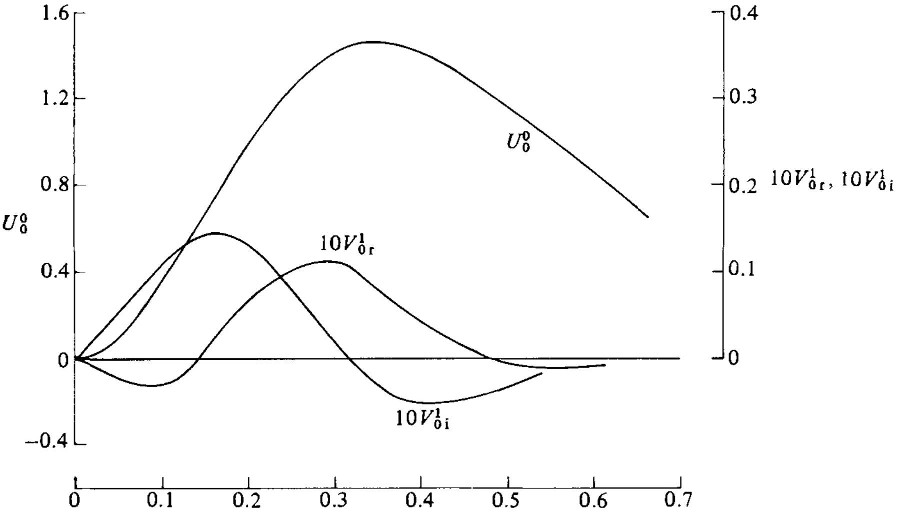

FIGURE 2. The eigenfunctions U_0^0 and V_0^1 corresponding to the critical case $T_0 = 11.99$ and $k = 0.51$.

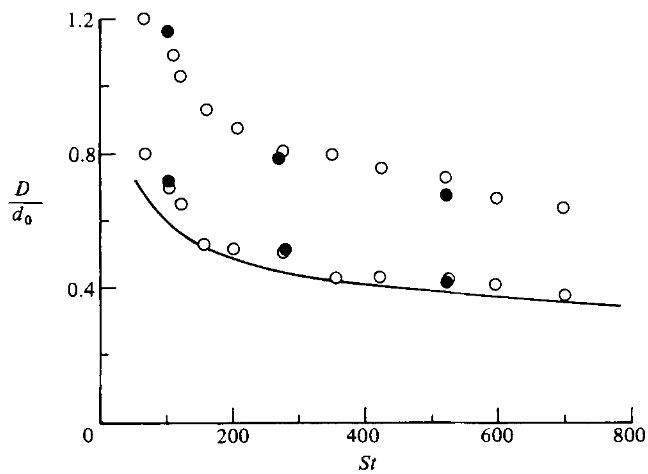

FIGURE 3. A comparison between Honji's experimental points and linear theory.

so that the critical value of R_s is

$$R_s = R_{sc} = 4.24[\beta^{\frac{1}{2}} + 0.46\beta^{\frac{1}{4}} + \dots]. \tag{3.2}$$

If R_s is greater than R_{sc} the vortices grow exponentially in time but remain localized near $\theta = \frac{1}{2}\pi$. In order to compare our result with those of Honji (1981) we rewrite (3.2) in the form

$$\lambda = \lambda_c = \frac{2.06}{\beta^{\frac{1}{4}}}\left[1 + \frac{0.23}{\beta^{\frac{1}{4}}} + \dots\right]. \tag{3.3}$$

We further note that, in the notation of Honji, λ is equal to the ratio of the cylinder oscillation amplitude d_0 to the diameter $D = 2a$ and that the Strouhal number St defined by Honji is related to β by

$$St = \frac{2}{\pi}\beta.$$

In figure 3 we have compared our theoretical prediction of $\lambda = d_0/D$ with Honji's results. We recall that above the lower of the two sequences of experimental points Honji observed Taylor–Görtler vortices. There seems little doubt that the instability mechanism discussed here is responsible for the vortices seen by Honji. Surprisingly we see that (3.3) is in excellent agreement with Honji's results even for $\lambda \sim 1$, even though (3.3) is formally valid only in the limit $\beta \to \infty$.

4. Linear theory for more general steady streaming flows

We shall in this section discuss the modifications to the expansion procedure of §3 that are necessary when the basic flow does not have the symmetry of the circular-cylinder problem. Suppose then that we consider the stability of the boundary layer induced by the outer potential flow $U_0 U(x) \cos \omega t$ interacting with a rigid wall of local radius of (convex) curvature $aR(x)$. Here x is a dimensionless variable which measures distance along the wall. We again take η to be a normal variable scaled on the Stokes-layer length scale $(2\nu/\omega)^{\frac{1}{2}}$, and the basic flow in the boundary layer will be of the form (2.3) with $\sin\theta$, $\cos\theta$ and $\sin 2\theta$ replaced by $U(x)$, $U'(x)$ and $\frac{1}{2}U(x)\,U'(x)$ respectively.

At any local station x along the wall the local Taylor number T_l which governs the stability of the boundary layer is given by

$$T_l = \frac{2^{\frac{3}{2}} U_0^2 U^2(x)}{a\nu^{\frac{1}{2}}\omega^{\frac{3}{2}} R(x)},$$

and instability of the localized type discussed in §3 will occur at $x = x_m$, which is a maximum of the function $U^2(x)\,R^{-1}(x)$. In the neighbourhood of x_m we write

$$\Phi = (x - x_m)\,\beta^{+\frac{1}{8}}$$

and expand the disturbance as in (2.8). The only essential differences are that the coefficients of the amplitude equation (2.14) are altered because (1) the radius of curvature must be expanded locally, and (2) the terms corresponding to the $\beta^{-\frac{1}{4}}$ terms in (2.3) remain $O(\beta^{-\frac{1}{4}})$ near $x = x_m$ and so contribute to the linear term in (2.14). However, the first-order term in the expansion of the Taylor number does not depend on these higher-order alterations, and so we can say that, correct to first order, the boundary layer is locally neutrally stable at $x = x_m$ if

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U^2}{R}\right) = 0, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{U^2}{R}\right) < 0 \quad \text{at} \quad x = x_m$$

and

$$T_l = 11.99 \quad \text{at} \quad x = x_m.$$

It is in fact more convenient to work with the overall non-local Taylor number $\bar{T}$ defined by

$$\bar{T} = \frac{2^{\frac{3}{2}} U_0^2}{a\nu^{\frac{1}{2}}\omega^{\frac{3}{2}}}.$$

We then conclude that the boundary layer is unstable to centrifugally driven vortices when

$$\bar{T} > 11.99/\mathrm{Max}\left(\frac{U^2(x)}{R(x)}\right), \tag{4.1}$$

where Max (U^2/R) denotes the largest maximum value of U^2/R. We see that, having performed the calculation of §3, the critical configuration defined by (4.1) can to first order be written down with a knowledge of only the outer potential flow and the local radius of curvature. We illustrate the simplicity of the procedure by considering the stability of the flow induced by the oscillation of an elliptic cylinder in a viscous fluid.

The basic flow induced by the high-frequency oscillation of an elliptical cylinder has been given by Davidson & Riley (1972). We suppose that the ellipse has major and minor axis of lengths $2a$ and $2b$ respectively and that the cylinder oscillates with velocity $U_0 \cos \omega t$ in a direction making an angle $-\alpha$ with the x-axis. In order to find the critical Taylor number for the flow, we require the first-order potential flow and the local radius of curvature of the ellipse. The potential flow is obtained in a routine manner by mapping the ellipse onto a circle of radius $\frac{1}{2}(a+b)$. If we use the parametric representation of the ellipse,

$$x = a\cos\phi, \quad y = b\sin\phi \quad (0 \leqslant \phi < 2\pi),$$

then the slip velocity of the potential flow is

$$U_0\, U(\phi) = \frac{U_0(1+K)\sin(\phi-\alpha)}{(\sin^2\phi + K^2\cos^2\phi)^{\frac{1}{2}}}, \quad \text{where} \quad K = \frac{b}{a}.$$

The radius of curvature is easily obtained in terms of ϕ, and we find that the local Taylor number T_1 is given by

$$T_1 = \frac{2^{\frac{3}{2}} U_0^2 (1+K)^2 \sin^2(\phi-\alpha)}{a\nu^{\frac{1}{2}}\omega^{\frac{3}{2}}(\sin^2\phi + K^2\cos^2\phi)^{\frac{5}{2}}},$$

and it follows that the flow is locally neutrally stable at $\phi = \phi_m$ if

$$\frac{dT_1}{d\phi} = 0, \quad T_1 = 11.99 \quad \text{at} \quad \phi = \phi_m.$$

We shall take a to be held fixed whilst K and the angle of attack α vary, and compare the critical Taylor number $\bar{T}$ for the flow with that appropriate to a circular cylinder of radius a.

The values of T_1 at which the flow is locally neutrally stable at some value of ϕ_m are found by considering the maxima of the function

$$S(\phi) = \sin^2(\phi-\alpha)(\sin^2\phi + K^2\cos^2\phi)^{-\frac{5}{2}}. \tag{4.2}$$

Unlike the circular-cylinder case there can be more than two values of ϕ_m at which this function has a maximum. In view of the symmetry of the problem we can restrict α to the range $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$ and ϕ to the range $0 \leqslant \phi < \pi$.

After some calculations it can be seen that the results for arbitrary values of α can be understood by first considering the case $\alpha = 0$. In this case the function $S(\phi)$ has a maximum at $\phi = \phi_1 = \frac{1}{2}\pi$ for all values of K. However, when $K < \sqrt{\frac{3}{5}}$ two further maxima occur at

$$\phi = \phi_2 = \sin^{-1}\left(\frac{2K^2}{3(1-K^2)}\right)^{\frac{1}{2}} \quad \text{and} \quad \phi = \phi_3 = \pi - \phi_2.$$

These two extra maxima emanate from $\phi = \frac{1}{2}\pi$ when $K = \sqrt{\frac{3}{5}}$ and

$$S(\phi_2) = S(\phi_3) > S(\phi_1) \quad \text{for} \quad K < \sqrt{\tfrac{3}{5}}.$$

Hence for $K < \sqrt{\frac{3}{5}}$ there are six potentially unstable points on the cylinder, and the most unstable points are $\phi = \phi_2, \phi_3, \phi_2+\pi, \phi_3+\pi$. The migration of the most unstable point away from $\phi = \frac{1}{2}\pi$ is due to the relatively large increase in curvature away from $\phi = \frac{1}{2}\pi$ caused by increasing the eccentricity of the ellipse. We note that when $\alpha = 0$

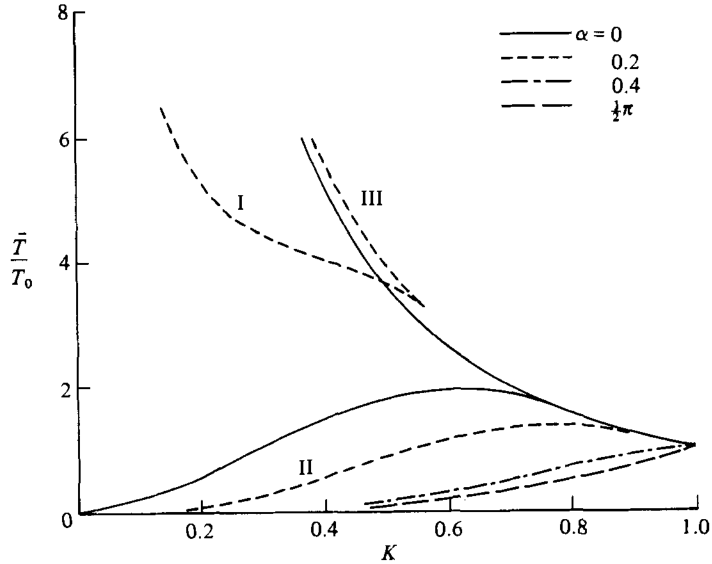

FIGURE 4. The dependence of $\bar{T}/\bar{T}_0$ on K for $\alpha = 0, 0.2, 0.4, \frac{1}{2}\pi$.

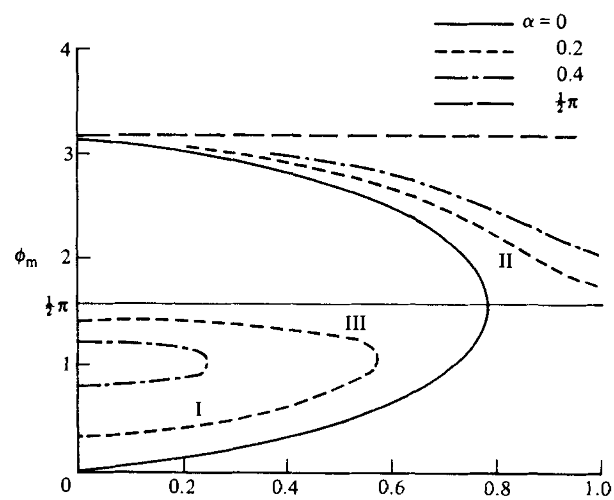

FIGURE 5. The dependence of ϕ_m on K for $\alpha = 0, 0.2, 0.4, \frac{1}{2}\pi$.

and $K = \sqrt{\frac{3}{5}}$ the scaling of §3 needs a more significant alteration since $\phi = \frac{1}{2}\pi$ is then a fourth-order turning point. We must then choose to work in a $\beta^{-\frac{1}{12}}$ neighbourhood of $\frac{1}{2}\pi$. The appropriate amplitude equation then has the term proportional to $\Phi^2 A$ in (2.3) replaced by $\Phi^4 A$.

In figure 4 we have shown the dependence of $\bar{T}/T_0$ (where T_0 is the critical value for the case $K = 1$) on K with $\alpha = 0$. We obtain a familiar cusp-shaped curve, and we note that, for $K > \sqrt{\frac{3}{5}}$, $\bar{T}/T_0$ is a single-valued function of K. The lower curve for $K < \sqrt{\frac{3}{5}}$ corresponds to the two equally unstable locations ϕ_2 and ϕ_3, and passes through the origin. This means that the critical value of $\bar{T}/T_0$ can be made arbitrarily small by taking the limit $K \to 0$. We can see in figure 5 that in this limit the locations of the most unstable positions approach $\phi = 0, \pi$, where the radius of curvature is clearly greatest.

The results for $\alpha \neq 0$ are obtained by describing the unfolding of figures 4 and 5 when $0 < \alpha \ll 1$. The upper and lower curves to the left of the cusp move up and to the left when α increases from zero. The lower curve for $\alpha = 0$ is in fact two coincident curves corresponding to $\phi = \phi_2$ and $\phi = \phi_3$. The other one of these curves remains connected to ($\bar{T}/T_0 = 1$, $K = 1$) and ($\bar{T}/T_0$, $K = 0$), but moves downwards until $\bar{T}/T_0$ is eventually a monotonically increasing function of K on this branch. Ultimately the branch asymptotes to the curve $\bar{T}/T_0 = 4K^5/(1+K)^2$, which corresponds to $\alpha = \frac{1}{2}\pi$, whilst the detached upper branches rapidly move to the left and upwards when α increases. Finally, when $\alpha = \frac{1}{2}\pi$ there are only two maxima on the cylinder at $\phi = 0, \pi$. In figures 4 and 5 we have illustrated this process for a few values of α. The curves I, II and III of these two figures correspond.

Suppose now that we have an elliptical cylinder with K fixed and we require the most stable or unstable orientation of this ellipse in an oscillatory flow. It follows from figure 4 that if we wish to keep the flow stable then we choose $\alpha = 0$, whilst if we wish to set up an unstable flow then we take $\alpha = \frac{1}{2}\pi$. Next suppose that the angle of attack α is held fixed and K can be varied. We see from figure 4 that for some values of α there is range of values for K which give a flow more stable than that around a circular cylinder of radius a. The most pronounced effect of increasing eccentricity corresponds to the $\alpha = \frac{1}{2}\pi$ case. Here we see that changing say K from 1 to $\frac{1}{2}$ produces a decrease in the critical Taylor number by a factor of ≈ 20.

5. The nonlinear development of the instability for the circular-cylinder problem

We shall now describe how nonlinear effects alter the linear development of the instability described in §2. In the following discussion we use the terms 'fundamental', 'mean', 'first-harmonic' with reference to the z dependence of the disturbance. We recall that A, the Φ-dependent amplitude of the disturbance, satisfies (2.14) and that exponentially decaying solutions of (2.14) exist only for certain values of T_1, namely those given by (2.17). We shall choose a scaling for the amplitude of the disturbance in such a way that nonlinear effects modify (2.17). Smaller disturbances can then be considered by a further limiting procedure, whilst larger disturbances are probably not accessible to a weakly nonlinear stability theory. If we wish to obtain the linear structure of (2.14), we must expand

$$T = T_{0c} + \beta^{-\frac{1}{4}}\bar{T} + \dots, \tag{5.1}$$

where T_{0c} is the critical value of T_0.

It remains for us to specify the appropriate scale for the disturbance. This is a routine procedure and can be done by first assuming that the azimuthal disturbance velocity field is $O(\beta^{-\delta})$. The interaction of the fundamental with itself produces through the nonlinear terms of the Navier–Stokes equations first-harmonic and mean-azimuthal-velocity fields $O(\beta^{2\delta})$. The first-harmonic and mean-velocity fields then interact with the fundamental to reinforce the latter at $O(\beta^{3\delta})$, so that, in view of (5.1), we must choose $\delta = \frac{1}{8}$ in order that (2.14) should be modified by nonlinear effects.

In principle the expansion procedure outlined above is straightforward; however, the time dependence of the fundamental, mean, etc. leads to some technical difficulties which do not arise for steady flows. The major surprise of the nonlinear calculation is the effect of the instability on the mean flow. We recall that the mean flow has a small steady component with a double boundary-layer structure. We shall see that the instability also drives a steady mean flow which turns out to be comparable to that driven by the first-order oscillatory flow. This result is at first surprising, since it implies that the instability calculated by weakly nonlinear stability theory modifies at first order the basic flow which has produced it. This is not the case, since the instability is driven by the oscillatory part of the basic flow and not by the steady-streaming component. However, the outer steady-streaming boundary layer is now driven simultaneously by the basic oscillatory flow and the instability, so that the instability does indeed have an $O(1)$ effect on the steady streaming.

We now outline the details of the expansion procedure described above. The disturbance quantities U, V, W and P then expand as

$$U = \beta^{-\frac{1}{8}}U_0 \cos k_c z + \beta^{-\frac{1}{4}}[U_1 \cos k_c z + U_2 \cos 2k_c z] + \beta^{-\frac{3}{8}}[U_3 \cos k_c z + U_4 \cos 2k_c z + U_5 \cos 3k_c z + U_{M0}] + \dots, \quad (5.2a)$$

$$V = \beta^{-\frac{1}{8}}V_0 \cos k_c z + \beta^{-\frac{1}{4}}[V_1 \cos k_c z + V_2 \cos 2k_c z + V_{M0}] + \beta^{-\frac{3}{8}}[V_3 \cos k_c z + V_4 \cos 2k_c z + V_5 \cos 3k_c z + V_{M1}] + \dots, \quad (5.2b)$$

$$W = \beta^{-\frac{1}{8}}W_0 \sin k_c z + \beta^{-\frac{1}{4}}[W_1 \sin k_c z + W_2 \sin 2k_c z] + \beta^{-\frac{3}{8}}[W_3 \sin k_c z + W_4 \sin 2k_c z + W_5 \sin 3k_c z] + \dots, \quad (5.2c)$$

$$P = \beta^{-\frac{1}{8}}P_0 \cos k_c z + \beta^{-\frac{1}{4}}[P_1 \cos k_c z + P_2 \cos 2k_c z] + \beta^{-\frac{3}{8}}[P_3 \cos k_c z + P_4 \cos 2k_c z + P_5 \cos 3k_c z] + P_{M0} + \beta^{-\frac{1}{8}}P_{M1} + \beta^{-\frac{1}{4}}P_{M2} + \beta^{-\frac{3}{8}}P_{M3} + \dots, \quad (5.2d)$$

where apart from P_{M0}, P_{M1} and P_{M2}, which depend only on τ and Φ, the coefficients in these expansions are functions of τ, Φ and η. The functions P_{M0}, P_{M1} and P_{M2} are essentially pressure eigenfunctions needed to satisfy all the required conditions on the mean velocity field. (See DiPrima & Stuart (1975) for a discussion of the need for such eigenfunctions in centrifugal instability problems.)

It is now a straightforward procedure to substitute from (5.2) into (2.5) and successively equate like powers of $\beta^{-\frac{1}{8}}$. At order $\beta^{-\frac{1}{8}}$ we find that (U_0, V_0) satisfy the linear stability problem (2.10) with $T_0 = T_{0c}$ and $k = k_c$, so that

$$(U_0, V_0) = A(\Phi)\,(\hat{U}_0, \hat{V}_0),$$

where
$$\hat{U}_0 = \sum_{-\infty}^{\infty} e^{in\tau}\, U_0^n, \quad \hat{V}_0 = \sum_{-\infty}^{\infty} e^{in\tau}\, V_0^n.$$

Here the functions $\{U_0^n\}$ and $\{V_0^n\}$ satisfy (2.12) with $k = k_c$ and $T_0 = T_{0c}$. At order $\beta^{-\frac{1}{4}}$ we find that terms proportional to $\sin k_c z$, $\cos k_c z$ again satisfy (2.13a), so that (U_1, V_1) is given by (2.13b). In addition to the fundamental modes generated at this order, there are first-harmonic and mean-flow correction terms produced by the nonlinear terms Q_1, Q_2 and Q_3 appearing in (2.5). After some manipulation we can show that (U_2, V_2) and (U_{M0}, V_{M0}) can be expressed in the forms

$$(U_2, V_2) = A^2(\Phi)\,(\hat{U}_2(\eta,\tau), \hat{V}_2(\eta,\tau)), \quad (U_{M0}, V_{M0}) = A(\Phi)\,(A'(\Phi)\,\hat{U}_{M0}, A(\Phi)\,\hat{V}_{M0}),$$

where $(\hat{U}_2, \hat{V}_2)$ satisfy

$$\left.\begin{aligned}
&\left\{\frac{\partial^2}{\partial\eta^2} - 4k_c^2 - 2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2} - 4k_c^2\right\}\hat{U}_2 - 8k_c^2 T_{0c}\bar{v}_0 \hat{V}_2 \\
&\qquad\qquad = k_c^2 T_{0c}\hat{V}_0^2 + 2\{\hat{U}_0 \hat{U}_{0\eta\eta\eta} - \hat{U}_{0\eta}\hat{U}_{0\eta\eta}\}, \\
&\left\{\frac{\partial^2}{\partial\eta^2} - 4k_c^2 - 2\frac{\partial}{\partial\tau}\right\}\hat{V}_2 - 4\frac{\partial}{\partial\eta}\bar{v}_0 \hat{U}_2 = \{\hat{U}_0 \hat{V}_{0\eta} - \hat{U}_{0\eta}\hat{V}_0\}, \\
&U_2 = V_2 = \frac{\partial}{\partial\eta}\hat{U}_2 = 0 \quad \text{at} \quad \eta = 0, \\
&\hat{U}_2, \hat{V}_2 \to 0 \quad \text{as} \quad \eta \to \infty.
\end{aligned}\right\} \tag{5.3}$$

The mean-flow correction term V_{M0} satisfies

$$\left.\begin{aligned}
\frac{\partial^2}{\partial\eta^2}(V_{M0}) - 2\frac{\partial V_{M0}}{\partial\tau} &= \frac{\partial}{\partial\eta}\{\hat{U}_0 \hat{V}_0\}, \\
V_{M0} = 0 \quad \text{for} \quad \eta &= 0, \infty.
\end{aligned}\right\} \tag{5.4}$$

The forcing term on the right-hand side of the equation for V_{M0} has the property that

$$\int_0^{2\pi} \hat{U}_0 \hat{V}_0 \,\mathrm{d}\tau = 0,$$

so that $\hat{V}_{M0}$ tends to zero exponentially when $\eta \to \infty$. The dependence of V_{M0} on the slow variable Φ induces the normal velocity component $U_{M0} = AA'\hat{U}_{M0}$. From the equation of continuity we have

$$\hat{U}_{M0} = -2^{-\frac{7}{4}} T_{0c} \int_0^\eta \hat{V}_{M0}\,\mathrm{d}\eta, \tag{5.5}$$

which tends to a function of τ when $\eta \to \infty$. Thus there exists a weak outer potential flow driven by the mean-velocity field in the Stokes layer. The outer potential flow decays algebraically in the normal direction and has a normal velocity that matches with (5.5). However, the matching requires an azimuthal velocity field of order $\beta^{-\frac{5}{8}}$ in the Stokes layer which tends to a given function of τ and Φ when $\eta \to \infty$. This velocity field is an 'eigensolution' driven by the pressure field P_{M0}, which is then determined by matching with the outer potential flow. This outer flow is determined by first noting that, when $\eta \to \infty$, U_{M0} can be written as

$$U_{M0} = A(\Phi)\,A'(\Phi) \sum_{-\infty}^{\infty} \beta_n \mathrm{e}^{\mathrm{i}n\tau}, \tag{5.6}$$

where $$\beta_n = -2^{-\frac{7}{4}} T_{0c} \int_0^{2\pi}\int_0^\infty \hat{V}_{M0} \mathrm{e}^{-\mathrm{i}n\tau}\,\mathrm{d}\eta\,\mathrm{d}\tau \quad (n = \pm 1, \pm 3, \pm 5, \ldots),$$

and $$\beta_n = 0 \quad (n = 0, \pm 2, \pm 4, \ldots).$$

In order to determine the outer flow, we write

$$\zeta = \frac{r-a}{a}\beta^{\frac{1}{8}},$$

so that the radial and azimuthal variations are now on the same lengthscale. The radial and azimuthal velocity components then expand as

$$\left.\begin{aligned} U &= \beta^{-\frac{3}{8}}(\nu\omega)^{\frac{1}{2}}\left\{\sum_{-\infty}^{\infty}\frac{\partial\Psi_n}{\partial\Phi}\mathrm{e}^{\mathrm{i}n\tau}+\dots\right\},\\ V &= -\beta^{-\frac{3}{8}}(\nu\omega)^{\frac{1}{2}}\left\{\sum_{-\infty}^{\infty}\frac{\partial\Psi_n}{\partial\zeta}\mathrm{e}^{\mathrm{i}n\tau}+\dots\right\},\end{aligned}\right\} \tag{5.7}$$

where $\Psi_n = 0$ for n even, whilst for n odd Ψ_n satisfies

$$\left.\begin{aligned} &\frac{\partial^2\Psi_n}{\partial\zeta^2}+\frac{\partial^2\Psi_n}{\partial\Phi^2} = 0,\\ &\Psi_n \to 0 \quad \text{as} \quad \zeta\to\infty,\\ &\Psi_n \to 0 \quad \text{as} \quad |\Phi|\to\infty,\\ &\Psi_n = \tfrac{1}{2}\beta_n A^2 \quad \text{at} \quad \zeta = 0.\end{aligned}\right\} \tag{5.8}$$

The solution of (5.8) is

$$\Psi_n = \frac{\zeta\beta_n}{\pi}\int_{-\infty}^{\infty}\frac{A^2(\theta)\,\mathrm{d}\theta}{\zeta^2+(\Phi-\theta)^2}. \tag{5.9}$$

The amplitude function $A(\Phi)$ is determined as a solvability condition on the differential system obtained by equating fundamental terms of order $\beta^{-\frac{3}{8}}$ after substituting for U, V, W and P from (5.2) into (2.5). The required condition is

$$\frac{\mathrm{d}^2A}{\mathrm{d}\Phi^2}+\mu(\overline{T}-\Phi^2)\,A = \gamma A^3, \tag{5.10}$$

where μ is given by (2.15) with $k = k_{\mathrm{c}}$ and $T_0 = T_{0\mathrm{c}}$. The constant γ is defined by

$$\gamma = \frac{-\displaystyle\int_0^{2\pi}\int_0^{\infty}\{U^+P+V^+Q\}\,\mathrm{d}\eta\,\mathrm{d}\tau}{2^{\frac{5}{4}}T_{0\mathrm{c}}^{\frac{1}{2}}\displaystyle\int_0^{2\pi}\int_0^{\infty}\left[\bar{v}_0\,V^+\hat{V}_1-\bar{v}_{0\eta\eta}\,U^+\hat{U}_1+\bar{v}_0\,U^+\left(\frac{\partial^2}{\partial\eta^2}-k_{\mathrm{c}}^2\right)\hat{U}_1\right]\mathrm{d}\eta\,\mathrm{d}\tau},$$

where

$$\begin{aligned} P = \tfrac{1}{2}k_{\mathrm{c}}^2T_{0\mathrm{c}}[\hat{V}_0\hat{V}_2+2\hat{V}_{\mathrm{M}0}\hat{V}_0]-3k_{\mathrm{c}}^2[\tfrac{1}{2}\hat{U}_0\hat{U}_{2\eta}+\hat{U}_2\hat{U}_{0\eta}]&\\ -[\hat{U}_2\hat{U}_{0\eta\eta\eta}+\tfrac{1}{2}\hat{U}_{2\eta}\hat{U}_{0\eta\eta}-\hat{U}_{0\eta}\hat{U}_{2\eta\eta}-\tfrac{1}{2}\hat{U}_0\hat{U}_{2\eta\eta\eta}]&\end{aligned}$$

and

$$Q = [\hat{U}_0\hat{V}_{2\eta}+2\hat{U}_0\hat{V}_{\mathrm{M}0\eta}+\hat{U}_2\hat{V}_{0\eta}+2\hat{V}_2\hat{U}_{0\eta}+\tfrac{1}{2}\hat{U}_{2\eta}\hat{V}_0].$$

The amplitude equation (5.10) must be solved subject to the condition

$$A\to 0 \quad \text{as} \quad |\Phi|\to\infty,$$

and of course reduces to (2.14) for $A \ll 1$. We postpone a discussion of the solution of (5.10) until after an investigation of the effect of a finite-amplitude solution on the steady streaming of the basic flow.

The fundamental terms of order $\beta^{-\frac{3}{8}}$ in (5.2) can be calculated when the solvability condition (5.10) is satisfied. The equations for the first- and second-harmonic functions of order $\beta^{-\frac{3}{8}}$ can be solved directly without recourse to solvability condition. The radial mean-flow function $U_{\mathrm{M}0}$ is determined by (5.5), so that at order $\beta^{-\frac{3}{8}}$ it remains for us to discuss the azimuthal mean-flow function V_{M1}. This function satisfies the equation

$$\frac{\partial^2}{\partial \eta^2} V_{M1} - 2\frac{\partial}{\partial \tau} V_{M1} = A(\Phi)\frac{dA(\Phi)}{d\Phi}\left\{\frac{\partial}{\partial \eta}(\hat{U}_0 \hat{V}_1 + \hat{U}_1 \hat{V}_0) + 2^{-\frac{3}{4}} T_{0c} \hat{V}_0^2\right\}, \tag{5.11}$$

which is to be solved subject to

$$V_{M1} = 0 \quad \text{for} \quad \eta = 0, \infty. \tag{5.12}$$

However, the form of the nonlinear terms in (5.11) means that V_{M1} has a steady term in its Fourier-series expansion. If the steady part of V_{M1} is denoted by V_{M10} then the appropriate boundary conditions for V_{M10} are

$$V_{M10} = 0 \quad \text{at} \quad \eta = 0, \qquad \frac{\partial}{\partial \eta} V_{M10} \to 0 \quad \text{as} \quad \eta \to \infty. \tag{5.13}$$

Thus the steady flow in the boundary layer induced by the finite-amplitude disturbance does not decay to zero when $\eta \to \infty$. In fact we see from (5.11) and (5.12) that when $\eta \to \infty$

$$V_{M1} \sim dA\frac{dA}{d\Phi},$$

where d is a constant to be calculated numerically. We found that $d = -4.39$, so that the azimuthal velocity component of the disturbance tends to $-4.39 U_0 \beta^{-\frac{3}{8}} A\, dA/d\Phi$ when $\eta \to \infty$. It is known (see Stuart 1966) that the steady part of the azimuthal velocity component of the basic flow tends to $3.2^{-\frac{3}{4}} T_{0c}^{\frac{1}{2}} U_0 \beta^{-\frac{3}{8}} \Phi$ when $\eta \to \infty$ with $\theta - \frac{1}{2}\pi = \phi\beta^{-\frac{1}{8}}$. We see then that the steady streaming of the two-dimensional flow is modified by the instability. Moreover, it follows that in the outer steady-streaming boundary layer the steady part of the basic flow and the instability cannot be found independently. This outer layer is of thickness $a\beta^{-\frac{1}{4}}$, and if we take the variable $\xi = [(r-a)/a]\beta^{\frac{1}{4}}$ we look for an outer steady flow given to first order by

$$v = \frac{\nu}{a}\beta^{\frac{3}{8}}\Psi_\xi, \quad u = -\frac{\nu}{a}\beta^{\frac{1}{4}}\Psi_\phi,$$

where Ψ satisfies

$$\Psi_{\xi\xi\xi\xi} = \Psi_\xi \Psi_{\xi\xi\Phi} - \Psi_\Phi \Psi_{\xi\xi\xi}, \tag{5.14}$$

which must be solved subject to

$$\left.\begin{aligned} &\Psi = 0 \quad \text{at} \quad \xi = 0, \\ &\Psi_\xi \to 0 \quad \text{as} \quad \xi \to \infty, \\ &\Psi_\xi \to T_{0c}\left[\frac{3\Phi}{2^{\frac{3}{2}}} - 2^{\frac{3}{4}} \times 4.39 T_{0c}^{-\frac{1}{2}} A \frac{dA}{d\Phi}\right] \quad \text{as} \quad \xi \to 0. \end{aligned}\right\} \tag{5.15}$$

If we set $A = 0$ above we obtain the equations governing the attachment of a steady-streaming boundary layer within a $\beta^{-\frac{1}{8}}$ neighbourhood of $\theta = \frac{1}{2}\pi$. In that case (5.14) is solved subject to

$$\Psi_\xi = 0 \quad \text{at} \quad \Phi = 0, \tag{5.16}$$

and the symmetry of (5.10) about $\Phi = 0$ means that (5.16) can still be applied since either A or $dA/d\Phi = 0$ at $\Phi = 0$. We note that for large Φ the condition (5.15) reduces to $\Psi_\xi \to T_{0c} 3\Phi/2^{\frac{3}{2}}$ as $\xi \to 0$, so that, assuming that the boundary layer remains attached for finite values of Φ, the extra term proportional to $A\, dA/d\Phi$ in (5.15) merely produces an origin shift in the large-Φ asymptotic solution of (5.14). The linear

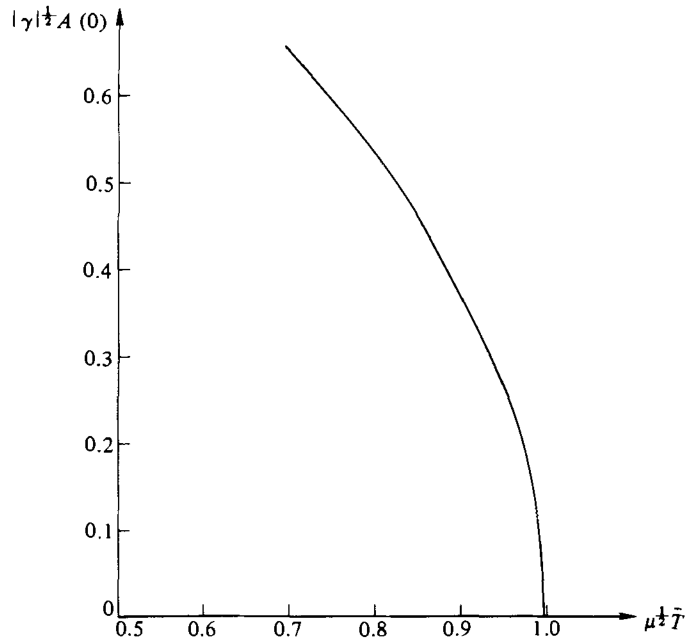

FIGURE 6. The numerically calculated bifurcating solution of (5.10).

eigenfunctions $A_n(\Phi)$ for $n \neq 0$ all have intervals where $A_n(\Phi)\,A_n'(\Phi)$ is positive, so that the possibility exists that in the nonlinear regime the slip velocity in (5.15) might change sign at some value of Φ. If the magnitude of the inviscid slip velocity is sufficiently large where this occurs then the attached-flow strategy fails and the steady-streaming boundary layer will detach prematurely from the cylinder. This possibility does not occur for more general flows where the point of attachment of the steady-streaming layer and the most unstable position do not coincide. In this case the steady streaming driven by the instability is weak compared with that of the basic flow.

We return now to discuss the solution of (5.10), which of course depends crucially on the sign of γ. Our calculations gave the totally unexpected result that

$$\gamma = -0.087.$$

This result is surprising, because, as we shall see below, it implies that the finite-amplitude solutions of (5.10) bifurcate subcritically from the eigenvalues of the linear problem. In the work of Seminara & Hall (1976) the corresponding constant was found to be positive, so that his sign difference caused some concern. The calculations used to compute γ were repeated using an independent program, and confirmed the result of the previous calculation. However, a completely independent calculation of γ would be desirable. The sign of γ could also be checked by a fully numerical investigation of the nonlinear problem, but the complicated nature of the basic flow makes such a calculation formidable.

On the assumption that γ is negative we then show from (5.10) that for $|\bar{T}-T_{1n}| \ll 1$

$$A^2 \sim \frac{\gamma}{\mu}(\bar{T}-T_{1n})\,A_n^2(\Phi)\frac{\displaystyle\int_{-\infty}^{\infty} A_n^2(\Phi)\,\mathrm{d}\Phi}{\displaystyle\int_{-\infty}^{\infty} A_n^4(\Phi)\,\mathrm{d}\Phi}, \qquad (5.17)$$

so that since $\mu > 0$ finite-amplitude solutions of (5.10) exist locally near $\bar{T} = T_{1n}$ only for $\bar{T} < T_{1n}$. Thus, as stated above, the solutions of (5.10) bifurcate subcritically from the eigenvalues of the linear problem. The subcritical nature of the bifurcation was confirmed by integrating (5.10) numerically using a shooting procedure. The results are shown in figure 6, where we have plotted the amplitude of the first mode evaluated at $\Phi = 0$ as a function of $\bar{T}$. In some hydrodynamic stability problems (see e.g. DiPrima & Sjbrand 1983) higher-order nonlinear effects can reverse this result and produce supercritical equilibrium solutions, though, without doing a higher-order calculation, we do not know that this happens in our problem.

It is well-known that a subcritical solution will be unstable, so that by allowing A to be a function of a slow time variable we can show that (5.17) is unstable. Thus in order to find the flow to which the disturbed flow evolves it is necessary to solve the fully nonlinear problem numerically; such a calculation has not yet been attempted.

6. Conclusions

We have shown that oscillatory viscous flows interacting with rigid boundaries of convex curvature can become unstable to Taylor–Görtler vortices. In particular, the flow induced by the transverse oscillations of a circular cylinder is linearly unstable to Taylor–Görtler vortices localized where the slip velocity of the potential flow outside the boundary layer on the cylinder is a maximum. The results of our theory are in excellent agreement with Honji's (1981) observations over a wide range of values of the frequency parameter β, even though our results are formally valid only in the limit $\beta \rightarrow \infty$.

For an elliptical cylinder there are as yet no experimental results available. It would be interesting to see whether the cusp-shaped curve for $\alpha = 0$ in figure 4 could be found experimentally. There is no reason to suppose that the sensitive dependence of the critical Taylor number on the eccentricity and the angle of attack predicted in §4 could not be reproduced experimentally.

The results of our nonlinear calculations are unexpected, because of the prediction of the subcritical nature of the instability. It is almost invariably the case in the Taylor problem that the bifurcation to a Taylor-vortex flow is supercritical, but DiPrima & Sjbrand (1983) have found subcritical bifurcation when considering the flow between counter-rotating cylinders. In fact, if we do not restrict the wavenumber to be that corresponding to the minimum on the neutral curve, there will always be a finite band of wavenumbers where the Landau coefficient γ is negative in the steady Taylor problem. This band of wavenumbers lies to the left of the point on the neutral curve where the wavenumbers on the left- and right-hand branches are in the ratio 1:2. In the present problem the constant γ becomes singular where the neutral values of the wavenumbers are $\bar{k} = 0.34$ and $2\bar{k}$. In fact, near $\bar{k}$ the constant γ behaves like $-1/(k-\bar{k})$, so that to the left of $\bar{k}$ there is a finite range of values of $\bar{k}$ for which γ is positive. However, calculations show that the range of wavenumbers is only of length $O(10^{-1})$ and γ then becomes positive again.

If the instability is indeed subcritical then we presume that close to the critical Taylor number sufficiently large perturbations to the basic state will grow. It is possible that higher-order nonlinear effects eventually cause these perturbations to equilibrate, and that this is why Honji observes some kind of steady state with Taylor–Görtler cells. In fact, even if nonlinear effects are not stabilizing at higher order, then, because of the localized nature of the instability with the flow unstable in a $\beta^{-\frac{1}{8}}$ neighbourhood of the most susceptible positions of the boundary layer, we might expect that some periodicity along the cylinder would be observed. Indeed, it is known in parallel or nearly parallel-flow stability theory that Tollmien–Schlichting waves can be observed even though they are subcritically unstable. In the present problem, the subcritical nature of the bifurcation could be investigated by solving the full stability equations by Fourier-analysing in the z-direction and solving a large system of coupled nonlinear partial differential equations, but such a computation would be nontrivial.

Finally, we point out that perhaps Honji's results might in fact suggest that the instability does not develop supercritically in the manner usually found in the Taylor problem. We refer to the fact that Honji gave two experimentally determined curves: one representing the onset of 'streaked flow', and a higher curve above which the streak could be observed because the flow was then separated and turbulent. We saw in §5 that some finite-amplitude solutions of (5.10) would cause the steady-streaming boundary layer to separate prematurely. Thus our nonlinear calculations do in fact suggest an increasingly likely breakdown in the basic flow structure when the Taylor number is increased. Alternatively the separated flow observed by Honji could be simply the unsteady two-dimensional separation of the Stokes layer on the cylinder.

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